Reduction of Large Flexible Spacecraft Models Using Internal Balancing Theory

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A new and computationally simple method for flexible spacecraft model reduction is presented. Using an internal balancing approximation shown to be valid when natural damping is small, the method provides quantitative modal rankings with respect to disturbance environment, actuator authority, sensor observability, and performance objective. These rankings are used to select a reduced set of structural modes for controller design, and also to anticipate potential closed-loop performance and stability problems resulting from modal truncation. The method is demonstrated using a 54-mode spacecraft example.

I. Introduction

THE deployment of large space structures will require feedback control to meet stringent vibration, line-of-sight, jitter, and surface quality performance criteria. Flexible spacecraft control design requires mathematical models of plant dynamics, actuator/sensor locations, and disturbance environment, all defined with respect to performance objectives. These "evaluation models," obtained by finite-element analysis, can contain hundreds of structural modes known with varying degrees of accuracy. This paper addresses open-loop reduction of these models for the purpose of feedback control design.

Model reduction for control design is motivated by practical issues: hardware limitations, reduced computation power on-line, and robustness. In principle, an "optimal" control can be generated for a large model using modern control techniques, but this control is at least as complex as the model itself. When the "reduced controller-order" constraint is formally imposed, optimal control methods lose their attractive closed-form solutions. Therefore, a reduced-order "control design" model is selected using the same criteria as for the evaluation model: namely, to represent the system accurately with respect to the performance objectives.

The typical approach to structural model reduction is mode selection. The simplest, dominant-frequency selection,[1] ignores the fact that actuator and sensor placement, disturbances, and performance requirements also affect modal dominance. Modal-cost analysis[2,3] addresses the last two issues by prescribing modal rankings based upon relative contributions to a stochastic cost functional. It is shown[2] that the modal costs decouple asymptotically as damping approaches zero. The influence of actuator/sensor placement upon modal dominance is not treated, however, and the problem of feedback control coupling the disturbance into open-loop "undisturbed" modes remains.

Recently a general model-reduction approach based upon state selection from a grammian-balanced ("internally balanced") coordinate representation has been proposed.[4-6] For a given input/output configuration, this approach produces a balanced approximation to the large model by defining and retaining the "most controllable/observable" states. Open-loop application of this reduction method to problems involving two sets of inputs (disturbances and actuators) and outputs (regulated variables and sensors) is in general not possible, because a different coordinate basis is required for each input/output pair. However, results for lightly damped, single-input, single-output, flexible systems[7] have shown that modal truncation and balanced approximation are equivalent, asymptotically, as damping approaches zero. The implied asymptotic equivalence of modal and balanced coordinates extends to the multi-input multi-output case, and is fundamental to the present work.

This paper gives a systematic mode selection procedure for large lightly damped structural models, based upon the theory of "internal balancing."[4] The model-reduction/control-design problem is described in Sec. II. Section III briefly reviews internal balancing theory, develops an approximation for multi-input, multi-output, lightly damped structures, and investigates the approximation accuracy for practical problems. Similarities with recent, independently obtained, single-input, single-output results[8] are noted. Section IV attempts a physical interpretation of the results with respect to the control design issues, and proposes a new mode selection methodology for lightly damped structures. A complete design example is given in Sec. V.

II. Problem Statement

Consider an evaluation model containing structural dynamics in modal form

$$\ddot{\eta} + 2Z\Omega\dot{\eta} + \Omega^2\eta = \mathscr{B}u + \mathscr{D}w$$

$$z = \mathscr{M}_1\dot{\eta} + \mathscr{M}_2\eta \quad \text{(measurements)}$$

$$y = \mathscr{C}_1\dot{\eta} + \mathscr{C}_2\eta \quad \text{(regulated variables)}$$

$$J = \lim_{t\to\infty} E\{y^Ty\} \quad \text{(performance measure)} \qquad (1)$$

where $\eta \in \boldsymbol{R}^n$, $Z = \text{diag}\{\zeta_i\}$, $\Omega = \text{diag}\{\omega_i\}$, u is the control vector, and w is the disturbance vector. Defining

$$b_i \triangleq i\text{th row of } \mathscr{B}$$

$$d_i \triangleq i\text{th row of } \mathscr{D}$$

$$m_{ji} \triangleq i\text{th column of } \mathscr{M}_j$$

$$c_{ji} \triangleq i\text{th column of } \mathscr{C}_j$$

an equivalent state-space representation of Eq. (1) is

$$\dot{x} = Ax + Bu + Dw$$

$$z = Mx$$

$$y = Cx$$

$$J = \lim_{t\to\infty} E\{y^Ty\} \qquad (2)$$

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where

$$x = [\dot{\eta}_1 \quad \eta_1 \quad \dot{\eta}_2 \quad \eta_2 \quad \cdots \quad \dot{\eta}_n \quad \eta_n]^T$$

$$A = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix}, \qquad A_i = \begin{bmatrix} -2\zeta_i\omega_i & \omega_i^2 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1^T & 0 & b_2^T & 0 & \cdots & b_n^T & 0 \end{bmatrix}^T$$

$$D = \begin{bmatrix} d_1^T & 0 & d_2^T & 0 & \cdots & d_n^T & 0 \end{bmatrix}^T$$

$$M = [m_{11} \quad m_{21} \quad m_{12} \quad m_{22} \quad \cdots \quad m_{1n} \quad m_{2n}]$$

$$C = [c_{11} \quad c_{21} \quad c_{12} \quad c_{22} \quad \cdots \quad c_{1n} \quad c_{2n}]$$

Mode selection for control design requires evaluation of the relative importance of each mode to the control problem. The specific issues which influence this evaluation are: 1) modal fidelity, 2) controllability, 3) observability, 4) disturbance environment, and 5) performance objective. None of these criteria stands alone. For example, it makes little sense to retain the most accurately modeled mode if it is neither controllable nor observable in the measurements. Similarly, a highly controllable mode adds little to the controller performance if it is (strictly) unobservable in the sensors and in the performance. The problem is to find a reduced-order model which addresses each of these design issues such that controls designed for the reduced model perform well on the evaluation model.

III. Internal Balancing of Lightly Damped Structures

In general, balanced model reduction is state selection in a special coordinate system. Before proceeding with mode selection, this theory is reviewed.

Balanced Model Reduction

Consider a linear, time-invariant, asymptotically stable state-space system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{3}$$

having controllability grammian W_c^2 and observability grammian W_0^2 given by

$$W_c^2 = \int_0^\infty e^{At}BB^Te^{A^Tt}\,\mathrm{d}t \quad \text{or} \quad AW_c^2 + W_c^2A^T + BB^T = 0 \tag{4}$$

$$W_0^2 = \int_0^\infty e^{A^Tt}C^TCe^{At}\,\mathrm{d}t \quad \text{or} \quad A^TW_0^2 + W_0^2A + C^TC = 0 \tag{5}$$

Model (3) is *internally balanced* if

$$W_c^2 = W_0^2 = \Sigma^2$$

where

$$\Sigma^2 = \mathrm{diag}[\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2]$$

and

$$i < j \Rightarrow \sigma_i^2 > \sigma_j^2$$

The σ_i^2's are termed the "second-order modes" of Model (3). It has been shown[4] that any model of this form, Model (3), can be taken to internally balanced form using similarity transformations.

Next consider the balanced model partitioned as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u$$

$$y = [C_1 \quad C_2]x$$

$$\Sigma^2 = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2^2 \end{bmatrix}$$

with $\Sigma_1^2 = \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_k^2\}$ and $\Sigma_2^2 = \{\sigma_{k+1}^2, \ldots, \sigma_n^2\}$. The essence of balanced model reduction[4,6] is that if $\sigma_k^2 \gg \sigma_{k+1}^2$, then the input affects x_2 much less than it affects x_1, and the output is affected by x_2 much less than by x_1.

The internally balanced coordinate representation has a number of desirable properties with respect to model reduction. It is unique, to within a sign change on the basis vectors, provided the σ_i^2's are distinct. The σ_i^2's are similarity invariants of (A, B, and C). Most remarkably, any arbitrary subsystem is guaranteed to be asymptotically stable (subject to an additional restriction on the basis vectors in the nondistinct case[6]).

Application of balanced reduction to very large models is no easy task. Indeed, "transformation methods" are generally undesirable for large structural models because of computational problems and loss of a physically meaningful state vector. In the following, it is shown that the balanced coordinates are a special case of modal coordinates, and, thus, that no explicit change of basis is necessary, provided the damping is very small and the frequencies are sufficiently distinct.

Application to Lightly Damped Structural Models

Consider the state-space representation of a structural dynamics model in modal form

$$\begin{bmatrix} \ddot{\eta}_i \\ \dot{\eta}_i \end{bmatrix} = \begin{bmatrix} -2\zeta_i\omega_i & -\omega_i^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\eta}_i \\ \eta_i \end{bmatrix} + \begin{bmatrix} b_i \\ 0 \end{bmatrix}u; \qquad i = 1, \ldots, n$$

$$y = \sum_1^n [c_{1i} \quad c_{2i}]\begin{bmatrix} \dot{\eta}_i \\ \eta_i \end{bmatrix} \tag{6}$$

Theorem

For any modal subsystem of Model (6), the transformation

$$\begin{bmatrix} \dot{\eta}_i \\ \eta_i \end{bmatrix} = \left\{\frac{b_ib_i^T}{4\zeta_i\omega_i}\right\}^{1/2}\begin{bmatrix} 1 & 0 \\ 0 & \omega_i^{-1} \end{bmatrix}V_i\begin{bmatrix} \sigma_{1i}^{-1} & 0 \\ 0 & \sigma_{2i}^{-1} \end{bmatrix}q_i$$

where

$$\sigma_{1i}^2 \triangleq \left\{\frac{(b_ib_i^T)^{1/2}}{4\zeta_i\omega_i}\right\} \times \left[c_{1i}^Tc_{1i} + \omega_i^{-2}c_{2i}^Tc_{2i}\left[1 - 2\zeta_i\gamma_i + 2\zeta_i(1+\gamma_i^2)^{1/2}\right]\right]^{1/2}$$

$$\sigma_{2i}^2 \triangleq \left\{\frac{(b_ib_i^T)^{1/2}}{4\zeta_i\omega_i}\right\} \times \left[c_{1i}^Tc_{1i} + \omega_i^{-2}c_{2i}^Tc_{2i}\left[1 - 2\zeta_i\gamma_i - 2\zeta_i(1+\gamma_i^2)^{1/2}\right]\right]^{1/2}$$

$$V_i \triangleq \begin{bmatrix} v_{1i} & -v_{2i} \\ v_{2i} & v_{1i} \end{bmatrix}, \qquad v_{1i} \triangleq [(1+v_i)/2]^{1/2}, \quad v_{2i} \triangleq [(1-v_i)/2]^{1/2}$$

$$v_i \triangleq \mathrm{sgn}(\gamma_i)[1+\gamma_i^{-2}]^{-1/2} \qquad \gamma_i \triangleq \omega_ic_{2i}^Tc_{1i}/(c_{2i}^Tc_{2i}) - \zeta_i$$

produces the following internally balanced modal subsystem

$$\dot{q}_i = \omega_i\begin{bmatrix} -2\zeta_i v_{1i}^2 & -(1-2\zeta_i v_{1i}v_{2i})(\sigma_{1i}/\sigma_{2i}) \\ (1+2\zeta_i v_{1i}v_{2i})(\sigma_{2i}/\sigma_{1i}) & -2\zeta_i v_{2i}^2 \end{bmatrix} q_i + \left\{\frac{4\zeta_i\omega_i}{b_i b_i^T}\right\}^{1/2}\begin{bmatrix} \sigma_{1i}v_{1i}b_i \\ -\sigma_{2i}v_{2i}b_i \end{bmatrix} u$$

$$y_i = \left\{\frac{b_i b_i^T}{4\zeta_i\omega_i}\right\}^{1/2}\begin{bmatrix} \sigma_{1i}^{-1}\left[c_{1i}v_{1i}+\omega_i^{-1}c_{2i}v_{2i}\right]^T \\ \sigma_{2i}^{-1}\left[-c_{1i}v_{2i}+\omega^{-1}c_{2i}v_{1i}\right]^T \end{bmatrix}^T q_i$$

$$\Sigma_i^2 = \begin{bmatrix} \sigma_{1i}^2 & 0 \\ 0 & \sigma_{2i}^2 \end{bmatrix}$$

Proof

The proof follows from direct substitution of Σ_i^2 into Models (4) and (5).

Noting that $\sigma_{1i}^2 \approx \sigma_{2i}^2$ when $\zeta_i \ll 1$, define the block-diagonal transformation

$$\begin{bmatrix} \dot{\eta}_i \\ \eta_i \end{bmatrix} = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i/\omega_i \end{bmatrix} V_i q_i \tag{7}$$

where

$$\alpha_i \triangleq \left[b_i b_i^T/\left(c_{1i}^T c_{1i}+\omega_i^{-2}c_{2i}^T c_{2i}\right)\right]^{1/4} \tag{8}$$

and V_i, v_i, and γ_i are as defined above. Applying Transformation (7) to Transformation (6), the following model is obtained:

$$\dot{q}_i = \tilde{A}_i q_i + \tilde{B}_i u, \quad i=1,\ldots,n; \qquad y=\sum_1^n \tilde{C}_i q_i \tag{9}$$

where

$$\tilde{A}_i = \omega_i\begin{bmatrix} -2\zeta_i v_{1i}^2 & -(1-2\zeta_i v_{1i}v_{2i}) \\ (1+2\zeta_i v_{1i}v_{2i}) & -2\zeta_i v_{2i}^2 \end{bmatrix}$$

$$\tilde{B}_i = \alpha_i^{-1}\begin{bmatrix} v_{1i}b_i \\ -v_{2i}b_i \end{bmatrix}$$

$$\tilde{C}_i = \alpha_i\left[\left(c_{1i}v_{1i}+\omega_i^{-1}c_{2i}v_{2i}\right) \quad \left(-c_{1i}v_{2i}+\omega_i^{-1}c_{2i}v_{1i}\right)\right]$$

Main Result

1) If $\zeta_i \ll 1$, $i=1,\ldots,n$, then each modal subsystem of Model (9) is approximately internally balanced with

$$\Sigma_i^2 \approx \tilde{\Sigma}_i^2 = \sigma_i^2 I$$

$$\sigma_i^2 = (4\zeta_i\omega_i)^{-1}\left[b_i b_i^T\left(c_{1i}^T c_{1i}+\omega_i^{-2}c_{2i}^T c_{2i}\right)\right]^{1/2} \tag{10}$$

2) If $\max(\zeta_i,\zeta_j)\max(\omega_i,\omega_j)/|\omega_i-\omega_j| \ll 1$, $i \neq j$, then the entire Model (9) is approximately internally balanced with

$$\Sigma^2 \approx \tilde{\Sigma}^2 = \begin{bmatrix} \sigma_1^2 I & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 I \end{bmatrix} \tag{11}$$

Proof

1) The error in the balancing approximation for each modal subsystem is:

$$\frac{\|\Sigma_i^2-\tilde{\Sigma}_i^2\|}{\|\tilde{\Sigma}_i^2\|} = \left\|\begin{bmatrix} \sigma_{1i}^2-\sigma_i^2 & 0 \\ 0 & \sigma_{2i}^2-\sigma_i^2 \end{bmatrix}\right\| \Big/ \sigma_i^2$$

$$\leq \left|1-\left[1-\frac{2\zeta_i c_{2i}^T c_{2i}\left[\gamma_i+\left(1+\gamma_i^2\right)^{1/2}\right]}{\omega_i^2 c_{1i}^T c_{1i}+c_{2i}^T c_{2i}}\right]^{1/2}\right|$$

$$\leq \left|1-\left[1-\frac{2\zeta_i\left[|\gamma_i|+\left(1+\gamma_i^2\right)^{1/2}\right]}{1+\left(|\gamma_i|+\zeta_i\right)^2}\right]^{1/2}\right|$$

$$\leq 1-(1-3\zeta_i)^{1/2} \quad \text{provided} \quad \zeta_i < 1/3$$

2) Consider any two-mode subsystem of the transformed model. The controllability grammian for this subsystem is approximated by:

$$W_{c_{ij}} = \begin{bmatrix} \tilde{\Sigma}_i^2 & P_{ji}^T \\ P_{ji} & \tilde{\Sigma}_j^2 \end{bmatrix}$$

where P_{ji} satisfies

$$\tilde{A}_j P_{ji} + P_{ji}\tilde{A}_i^T + \tilde{B}_j\tilde{B}_i^T = 0$$

Solving for P_{ji} and taking the L_2-norm leads to

$$\|P_{ji}\| \leq 2\sigma_j\sigma_i(1/\rho_1+1/\rho_2) \times\left[\zeta_j\omega_j\zeta_i\omega_i/\{(1-\zeta_j)(1-\zeta_i)\}\right]^{1/2}$$

$$\rho_1,\rho_2 = \left[\omega_j^2+\omega_i^2+2\zeta_j\omega_j\zeta_i\omega_i \pm 2\omega_j\omega_i \times\left\{\left(1-\zeta_j^2\right)\left(1-\zeta_i^2\right)\right\}^{1/2}\right]^{1/2}$$

Removing the subscripts to denote the maximum, we obtain

$$\|P_{ji}\|/\sigma^2 \leq 2\zeta(\omega/\Delta\omega+1)/(1-\zeta) \approx 2(\zeta\omega/\Delta\omega+\zeta) \quad \text{if } \zeta \ll 1$$

from which it follows that

$$\frac{\|W_{c_{ij}}-\tilde{\Sigma}_{ij}^2\|}{\|\tilde{\Sigma}_{ij}^2\|} = \frac{\|P_{ji}\|}{\sigma^2} \ll 1 \quad \text{provided } (\zeta\omega/\Delta\omega) \ll 1$$

Similar results can be obtained for the observability grammian.

Rate Output Case

When $c_{2i}=0$, $i=1,\ldots,n$, Approximation (10) is exact and each modal subsystem of Model (6) is balanced, regardless of its damping ratio. To illustrate the approximation error in Approximation (11), consider the two-mode subsystem obtained from Model (6), assuming $\zeta_1=\zeta_2=\zeta$ and normalizing

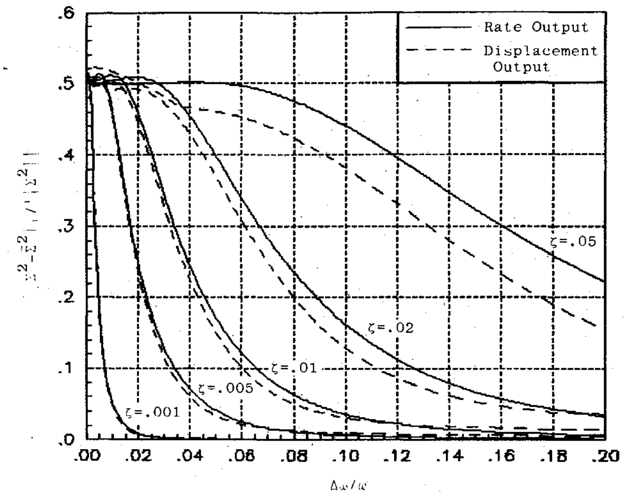

Fig. 1 Approximation error for two-mode subsystem.

the first frequency to unity

$$\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} = \left[\begin{array}{cc|cc} -2\zeta & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2\zeta(1+\Delta\omega/\omega) & -(1+\Delta\omega/\omega)^2 \\ 0 & 0 & 1 & 0 \end{array}\right] \times \begin{bmatrix} q_1 \\ \hline q_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ \hline b_2 \\ 0 \end{bmatrix} u$$

$$y_{12} = [c_{11} \quad c_{21} \; \vdots \; c_{12} \quad c_{22}] \begin{bmatrix} q_1 \\ \hline q_1 \end{bmatrix} \tag{12}$$

where $q' = \mathrm{d}q/\mathrm{d}\tau$ and $\tau = t\omega$. Choosing $b_1 = b_2 = c_{11} = c_{12} = 1$ the true (Σ) and approximate $(\tilde{\Sigma})$ balanced grammians are computed for several values of ζ and a range of $\Delta\omega$. The plot of approximation error in Fig. 1 shows the improvement in modal decoupling as damping ratio decreases. At $\zeta = 0.001$, for example, a 1% frequency difference results in only 5% approximation error.

Displacement Output Case

When $c_{1i} = 0$, $i = 1, \ldots, n$, the modal balancing error bound is

$$\frac{\|\Sigma_i^2 - \tilde{\Sigma}_i^2\|}{\|\tilde{\Sigma}_i^2\|} \leq \left| 1 - \left[1 - 2\zeta_i(1+\zeta_i^2)^{1/2} - 2\zeta_i^2 \right]^{1/2} \right|$$

$$\leq \zeta_i(2 + 2\zeta_i + \zeta_i^2)$$

When $\zeta_i < 0.1$, this bound is comparable to that given in Ref. 8 for the single-input, single-output case

$$4\zeta_i(1+2\zeta_i^2)^{1/2}(1+2\zeta_i)^{-1}(2+2\zeta_i^2)^{-1/2}$$

Choosing $b_1 = b_2 = c_{21} = c_{22}$ in Model (12), the displacement output balancing error is computed and plotted in Fig. 1.

IV. Mode Selection Methodology

The approximate modal balancing result enables quantitative modal analysis of the evaluation model with respect to physically meaningful design issues. Four modal rankings are discussed, followed by a suggested mode selection procedure.

Disturbance to Regulated Variables: Σ_{DC}^2

The $\{\sigma_{DC_i}^2\}$ gives a modal ranking in terms of open-loop performance. Large values indicate high disturbance propagation to the output, while small values imply low open-loop performance contribution. If the goal were simply to match open-loop performance, mode selection would be based on these rankings. Observe that the $\{\sigma_{DC_i}^2\}$ is equal to the balanced steady-state covariance when the model is forced with unit-intensity white noise. An interesting relationship exists between the $\{\sigma_{DC_i}^2\}$ and the modal costs[3] of Model (2). Assuming a zero-mean, unit-intensity white noise disturbance and light damping, the open-loop modal costs are approximated by

$$\mu_i = (4\zeta_i\omega_i^3)^{-1}\left[d_i d_i^T(\omega_i^2 c_{1i}^T c_{1i} + c_{2i}^T c_{2i})\right]$$

Using Eq. (10), the following relationship is obtained

$$\mu_i = 4\zeta_i\omega_i\sigma_i^4$$

Thus, modal-cost and internal-balancing criteria can differ. For example, if two modes have $\sigma_i = \sigma_j$ and $\omega_i > \omega_j$, then modal cost favors the higher frequency while balancing ranks them equally.

Actuators to Regulated Variables: Σ_{BC}^2

These rankings show modal controllability of the performance. A small value of $\sigma_{BC_i}^2$ indicates that the given actuator configuration has little direct effect upon the contribution of mode i to the performance, regardless of its open-loop performance contribution. In particular, if $\sigma_{BC_i}^2/\|\Sigma_{BC}^2\|$ is small and $\sigma_{DC_i}^2/\|\Sigma_{DC}^2\|$ is large, a redesign of the actuator configuration is suggested.

Disturbances to Sensors: Σ_{DM}^2

The $\{\sigma_{DM_i}^2\}$ shows modal observability of the disturbance in the sensors. A mode having a small relative $\sigma_{DM_i}^2$ may be impossible to estimate on-line. If the corresponding $\sigma_{DC_i}^2$ is large, the selection of sensor locations or types is inappropriate.

Actuators to Sensors: Σ_{BM}^2

The $\{\sigma_{BM_i}^2\}$ provides a modal analysis of potential controller authority for the given actuator and sensor configuration. Ideally the modes with large $\sigma_{BM_i}^2$ should align with those having large $\sigma_{DC_i}^2$ so that controller authority matches the performance objective. A mode with large $\sigma_{BM_i}^2$ should be included in the design model even if $\sigma_{DC_i}^2$ is low, particularly if it is in the controller bandwidth, to prevent spillover problems.

Mode-Selection Procedure

Using the four modal rankings, the mode-selection process is performed as follows:

1) Select the modes having the largest $\sigma_{DC_i}^2$. These modes contribute most to the performance objectives, and assuming reasonable actuator and sensor placements, it should be possible to control each of the modes to some extent.

2) Examine the $\{\sigma_{BM_i}^2\}$. Include in the design model any highly controllable/measurable modes not selected in Model (1), especially if they are close in frequency to selected modes. Omission of these modes can cause spillover, which can destabilize the system.

3) Examine the $\{\sigma_{DM_i}^2\}$ and the $\{\sigma_{BC_i}^2\}$. Unselected modes having large values in either of these rankings indicate actuator/sensor configuration pathologies. A large $\sigma_{DM_i}^2$ indicates an unmodeled mode in the measurements, which will inhibit state estimation. An unmodeled mode with large $\sigma_{BC_i}^2$ may be driven unpredictably by the controller to the detriment of performance. In either case, the modes should be included.

V. A Design Example

This application is performed on the Charles Stark Draper Laboratory Model 2 (Revision 3),[9] which has 54 flexible modes, each with 0.2% open-loop damping. Six disturbance inputs are modeled as independent colored-noise sources, each having a 15-Hz rolloff and a mean-squared value of 600π N^2. There are nine colocated force-actuator/rate-sensor pairs and an additional measurement of line-of-sight (LOS) pointing error. Open-loop modal frequencies are given in Table 1. The linear evaluation model has 114 states (108 structural and 6 disturbance).

This example follows the two-level "high-authority/low-authority" control design approach described in Ref. 10. The balanced mode-selection methodology is applied for control of LOS pointing performance (HAC), and "low-authority control" (LAC)[11] using colocated actuators and rate-sensors is appended to the control law to prevent high-frequency "spillover."[12]

Table 1 Open-loop evaluation model frequencies

Mode No.	Frequency, Hz	Mode No.	Frequency, Hz	Mode No.	Frequency, Hz
1	0.1131	19	2.2499	37	11.6523
2	0.1469	20	2.2541	38	11.6606
3	0.1490	21	3.4335	39	11.7353
4	0.1741	22	3.4522	40	13.3386
5	0.4549	23	3.9571	41	14.0969
6	0.5568	24	3.9861	42	14.8900
7	0.5953	25	4.0513	43	16.5092
8	0.6131	26	4.3367	44	17.1423
9	0.6351	27	6.5503	45	17.5072
10	0.6403	28	8.0570	46	17.7673
11	0.8151	29	8.4335	47	17.7682
12	0.8160	30	8.8400	48	17.7686
13	0.8235	31	8.9907	49	17.7767
14	0.9152	32	10.2474	50	21.0746
15	0.9703	33	10.5133	51	21.6642
16	1.1582	34	11.4797	52	22.0228
17	1.5508	35	11.6510	53	23.4744
18	1.7728	36	11.6522	54	23.8904

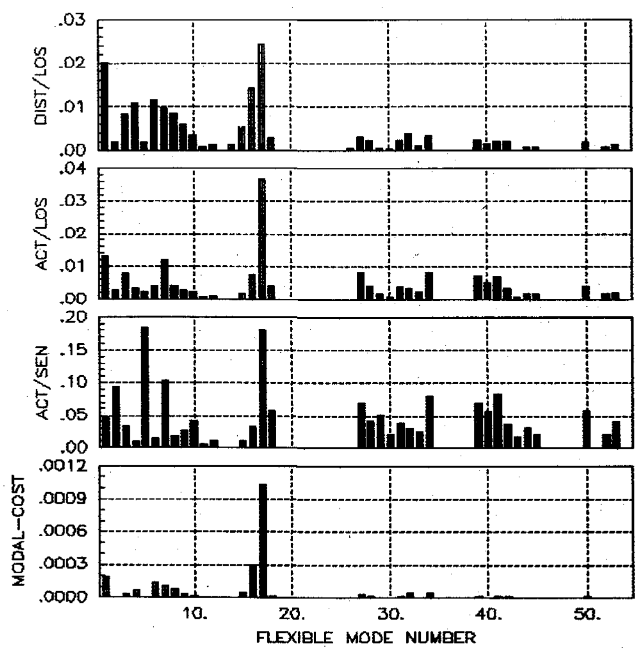

Fig. 2 Open-loop modal analysis.

Table 2 Modal rankings suggested by the analysis

DIST/LOS		ACT/LOS		ACT/SEN		Modal cost	
Mode	σ_i	Mode	σ_i	Mode	σ_i	Mode	$\mu_i^{1/2}$
17	0.0242	17	0.0366	5	0.1836	17	0.0010
1	0.0202	1	0.0130	17	0.1813	16	0.0003
16	0.0141	7	0.0119	7	0.1037	1	0.0002
6	0.0116	34	0.0081	2	0.0941	6	0.0001
4	0.0110	27	0.0080	41	0.0831	7	0.0001
7	0.0099	3	0.0078	34	0.0786	8	0.0001
8	0.0085	16	0.0074	27	0.0702	4	0.0001
3	0.0084	39	0.0072	39	0.0700	32	0.0001
9	0.0059	41	0.0066	18	0.0582	15	0.0000
15	0.0056	40	0.0051	50	0.0570	34	0.0000
32	0.0038	6	0.0042	40	0.0564	9	0.0000
34	0.0034	28	0.0042	29	0.0510	3	0.0000
10	0.0033	50	0.0041	1	0.0466	27	0.0000
27	0.0032	18	0.0039	28	0.0426	39	0.0000
18	0.0028	8	0.0038	10	0.0417	31	0.0000
31	0.0025	31	0.0038	53	0.0401	28	0.0000
39	0.0024	32	0.0034	31	0.0379	42	0.0000
28	0.0024	42	0.0034	42	0.0367	41	0.0000

Modal Analysis

Using the approximate internal balancing relation, Approximation (10), the following modal rankings are obtained: 1) Disturbances—LOS, 2) Actuators—LOS, and 3) Actuators—Rate Sensors.

Absolute values of the open-loop modal costs[3] are computed for comparison using the colored-noise disturbance. The rms second-order modes and modal costs are plotted vs mode number in Fig. 2. Immediately evident is the clustering of these modal phenomena. The disturbance effect as seen through the line-of-sight is constrained to clusters of modes, as is the ability to measure and control the model. Alignment of the "controllable clusters" and "disturbable clusters" indicates a favorable actuator/sensor configuration for the problem. Table 2 gives the quantitative modal ranking prescribed by each method.

Control Designs

Three controllers are designed and analyzed for the model. The first uses ten low-frequency modes from the model and relatively low gains, i.e., a cautious design. Based upon closed-loop analysis of this controller and re-evaluation of Fig. 2, a second controller is designed. The bandwidth of this controller is allowed to expand into an adjacent "dead zone" of the model under the assumption that insensitive modes cannot cause spillover. A low-authority controller[11] is designed and added to the high-gain controller to give the third design. Full closed-loop modal and stochastic analyses are presented.

Cautious Controller

Mode Selection

Using the modal analyses of Fig. 2, Modes 1, 3, 4, 6, 7, 8, 9, 15, 16, and 17 are selected. These are the first ten most disturbable modes with respect to LOS. Inspection of the second column of Table 2 shows that several controllable modes in the design model bandwidth have been omitted, and it is reasonable to expect that any undesirable in-band controller effects will be related to these modes.

Control Design

The ten selected modes and six disturbance states transform to a 26th order linear design model. Using standard LQG techniques, a state-feedback control is found which minimizes

$$J = \lim_{t \to \infty} E\left\{ z_R^T z_R + b u_R^T u_R \right\} \qquad (13)$$

Table 3 Closed-loop control spectrum of ten-mode model

Part			
Real	Imaginary	Frequency, Hz	Damping ratio
−4.8001	10.8240	1.8845	0.4054
−0.0270	7.3328	1.1671	0.0037
−0.0127	6.0956	0.9701	0.0021
−0.0115	3.9841	0.6341	0.0029
−0.0114	3.8621	0.6147	0.0030
−0.2178	3.7528	0.5983	0.0579
−0.0114	3.4864	0.5549	0.0033
−0.0033	1.0962	0.1745	0.0030
−0.0222	0.9388	0.1495	0.0237
−0.0203	0.7341	0.1169	0.0276

Table 4 Closed-loop filter spectrum of ten-mode model

Part			
Real	Imaginary	Frequency, Hz	Damping ratio
−5.5453	11.2239	1.9925	0.4430
−0.5170	7.3093	1.1662	0.0706
−0.0829	6.0898	0.9693	0.0136
−0.0654	3.9831	0.6340	0.0164
−0.2152	3.9445	0.6287	0.0545
−0.4231	3.6162	0.5795	0.1162
−0.0944	3.6258	0.5773	0.0260
−0.0877	1.0982	0.1753	0.0796
−0.0699	0.9299	0.1484	0.0750
−0.1559	0.7796	0.1265	0.1961

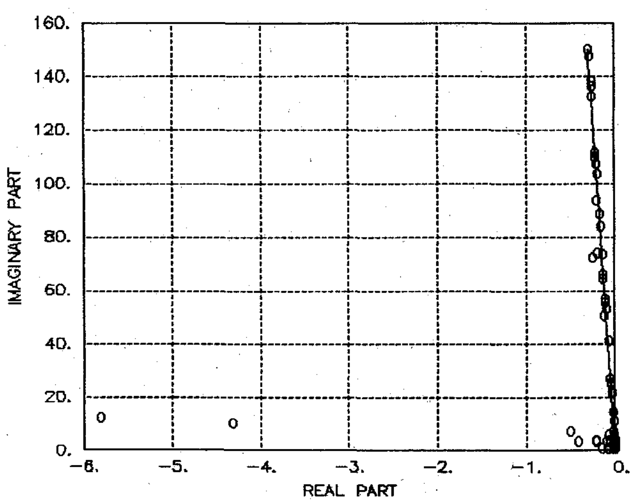

Fig. 3 Closed-loop spectrum with ten-mode controller.

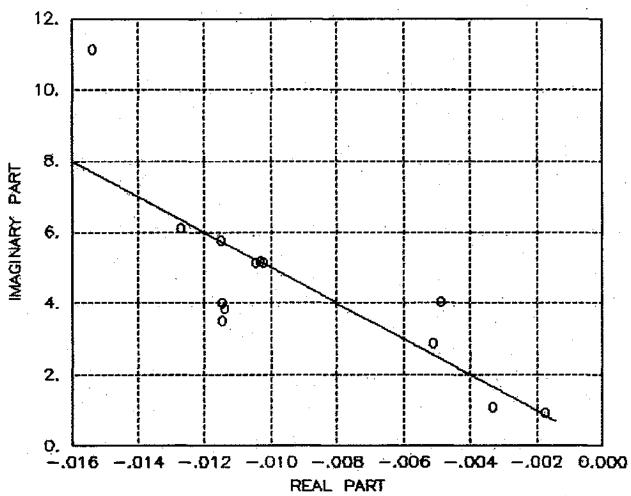

Fig. 4 Closed-loop spectrum with ten-mode controller (expanded scale).

Table 5 Stochastic performance analysis of controllers

	rss LOS x, μrad	rss LOS y, μrad	Total rss, μrad	rss control effort, N
Open-loop	131.7	1109.5	1117.3	0
Ten-mode control	62.4	175.9	186.6	12.2
14-mode control	69.0	87.5	111.4	28.6
14-mode control with LAC	26.4	75.6	80.1	25.3

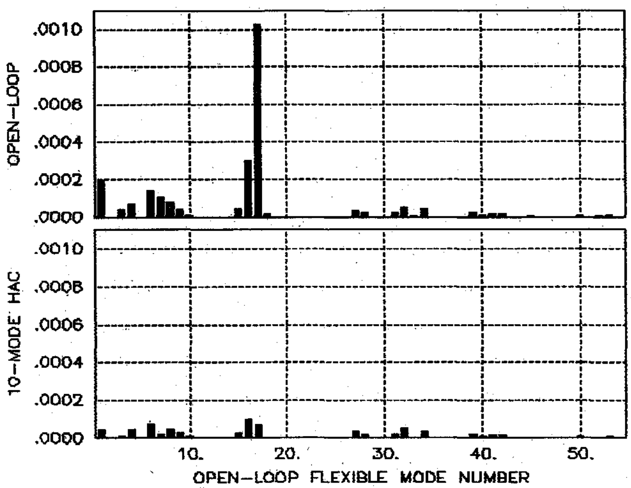

Fig. 5 Closed-loop modal costs for ten-mode controller.

where z_R is the LOS output of the reduced model and b is a tuning parameter. A reasonable closed-loop design spectrum, shown in Table 3, is obtained using $b = 10^{-11}$. The significant effect of this control is on Modes 7 and 17. Mode 7 damping increases by nearly a factor of 30, but its frequency increases by less than 1%. Mode 17 experiences a damping increase of a factor of 200 and a frequency increase of 21%, placing it between the frequencies of unmodeled Modes 18 and 19. Referring again to Fig. 2, Modes 19, 20, and 21 are apparently uncontrollable with respect to both line-of-sight and sensor placement. Mode 18, however, has some controllability (it ranks 14th out of 54). The control bandwidth is therefore likely to influence Mode 18.

A standard Kalman filter is found for the design model using the actual disturbance model and LOS "measurement noise" of intensity 5×10^{-10}, which gives a filter bandwidth roughly equivalent to the control bandwidth. The filter spectrum is shown in Table 4. Here, also, the frequency increase in Mode 17 is sufficient to encompass unmodeled Mode 18.

Evaluation

The ten-mode controller is implemented with the full 114th order model. Figure 3 shows that the closed-loop spectrum is stable, and also that the dominant control effects are limited to a few poles. The expanded scale in Fig. 4 shows the anticipated spillover in Mode 18. Notice also the decreased damping in unmodeled Modes 2, 5, and 10, which are well within the controller bandwidth. This "in-band spillover"

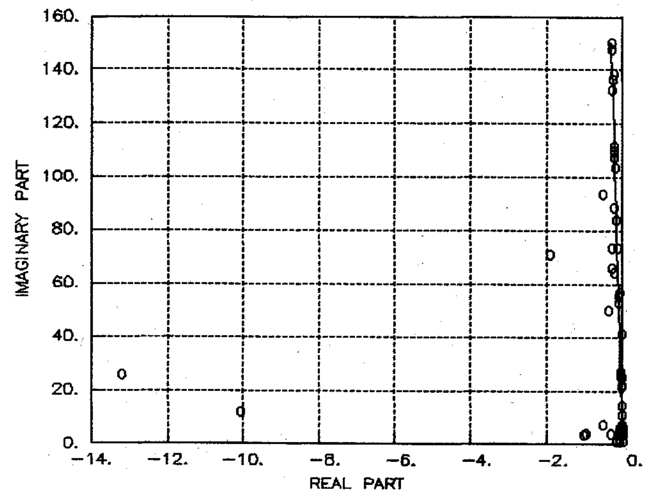

Fig. 6 Closed-loop spectrum with 14-mode controller.

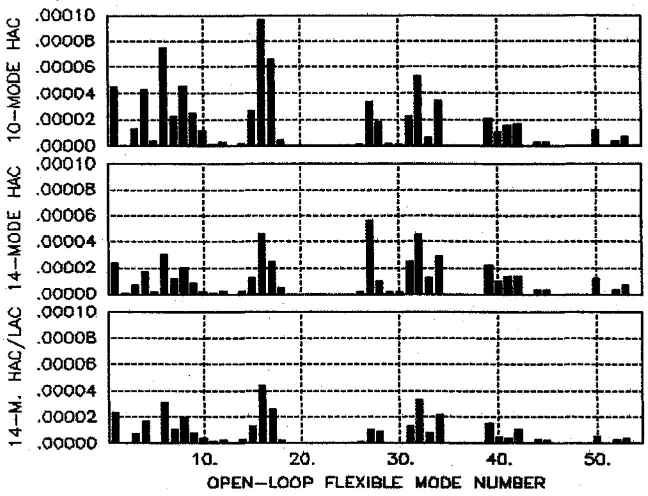

Fig. 8 Closed-loop modal costs for three controllers.

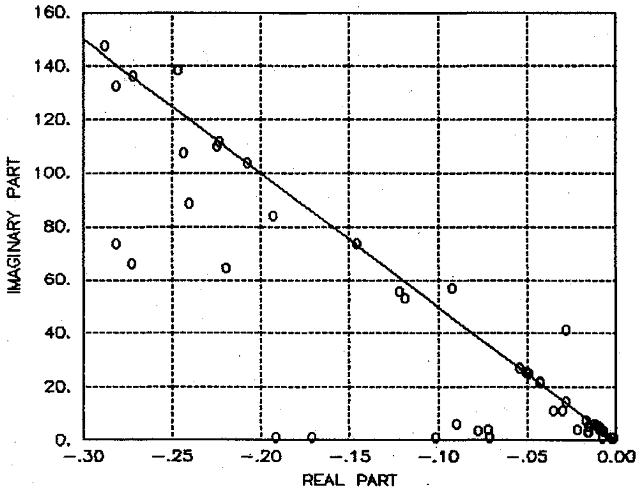

Fig. 7 Closed-loop spectrum with 14-mode controller (expanded scale).

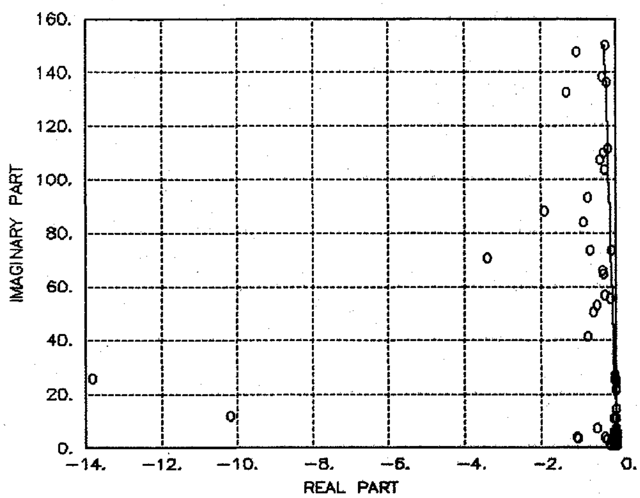

Fig. 9 Closed-loop spectrum for 14-mode controller with LAC.

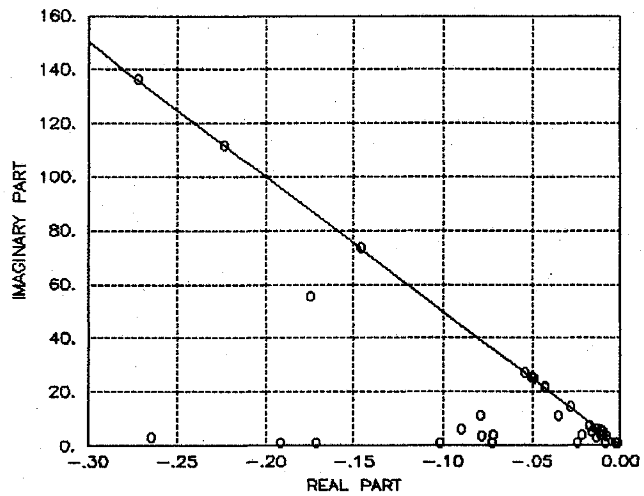

Fig. 10 Closed-loop spectrum for 14-mode controller with LAC (expanded scale).

occurs despite the relatively low controller authority over these three modes (see Fig. 2).

The stochastic performance with this controller is shown in Table 5. Using a total rss control effort of 12.2 N, the rss LOS error is reduced to 16% of its open-loop value. Figure 5 shows the absolute rms modal costs with and without this controller. Significant cost-level reductions occur in Modes 17, 16, and 1, which are the largest open-loop contributors. Notice that the cost contributions of "spillover" Modes 2, 5, 10, and 18 do not increase.

High-Gain Controller

Mode Selection

To account for the observed spillover effects and improve closed-loop performance, Modes 2, 5, and 10, and high-frequency Mode 18 are added to the control design model. The fact that Modes 19-26 are highly insensitive to control and disturbance inputs (see Fig. 2) is used to justify a closed-loop controller bandwidth extending into this dead zone.

Control Design

As before, a standard linear LOS regulator and Kalman filter is designed. For a control penalty of $5\times10^{-12}I$ and "measurement noise" of $5\times10^{-11}I$, the controller has bandwidth to 3.5 Hz which is well into the dead zone.

Evaluation

The closed-loop spectrum resulting from the high-gain control is shown in Fig. 6. As before, the dominant control effects are limited to a few modes. The expanded scale in Fig. 7 shows that the inclusion of Modes 2, 5, and 10 in the design model eliminates their in-band spillover. The expected high-frequency spillover is present, particularly in Mode 27, which is the closest unmodeled mode to the controller bandwidth having significant controllability (see Fig. 2). Notice that, as expected, no problems occur in the dead zone modes. Higher frequency spillover effects are evident in Modes 53, 52, and 31, which are well above the bandwidth of model certainty and cannot be handled explicitly. Figure 8 shows the modal costs of the 14-mode control compared with the ten-mode control. Notice the overall improvement in performance, with the exception of a spillover-induced cost increase in Mode 27. The total rss LOS error with this controller is 1% of the open-loop value (see Table 5).

High-Gain Controller with Low-Authority Control

The purpose of low-authority control (LAC) is to add damping to structural modes in the high-frequency uncertainty region. This method is described fully in Ref. 11. The actuator to rate sensor rankings in Fig. 2 indicate potential LAC modal authority. For this example, an LAC is designed to add 4% damping to the structural modes.

Evaluation

Figure 9 shows the characteristic damping increase in the closed-loop spectrum. The expanded scale in Fig. 10 and the modal costs in Fig. 8 show that all spillover effects, including Mode 27, are suppressed. The stochastic performance is included in Table 5.

VI. Summary and Conclusions

An approximation to the internally balanced coordinate representation for lightly damped structural modes is derived. It is shown that the balancing transformation preserves the block-modal structure of the model, enabling detailed quantitative modal analyses with respect to controllability, observability, disturbability, and performance, without a change of coordinates. Using these analyses, a methodology for mode selection is proposed and successfully applied to a large spacecraft model.

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